

# The Last Digit of $\binom{2n}{n}$ and $\sum \binom{n}{i} \binom{2n-2i}{n-i}$

Walter Shur  
20 Speyside Circle  
Pittsboro, NC 27312  
wrshur@gmail.com

## Abstract

Let  $f_n = \sum_{i=0}^n \binom{n}{i} \binom{2n-2i}{n-i}$ ,  $g_n = \sum_{i=1}^n \binom{n}{i} \binom{2n-2i}{n-i}$ . Let  $\{a_k\}_{k=1}$  be the set of all positive integers  $n$ , in increasing order, for which  $\binom{2n}{n}$  is not divisible by 5, and let  $\{b_k\}_{k=1}$  be the set of all positive integers  $n$ , in increasing order, for which  $g_n$  is not divisible by 5. This note finds simple formulas for  $a_k$ ,  $b_k$ ,  $\binom{2n}{n} \bmod 10$ ,  $f_n \bmod 10$ , and  $g_n \bmod 10$ .

## Definitions

$$f_n = \sum_{i=0}^n \binom{n}{i} \binom{2n-2i}{n-i}; \quad g_n = \sum_{i=1}^n \binom{n}{i} \binom{2n-2i}{n-i}$$

$\{a_k\}_{k=1}$  is the set of all positive integers  $n$ , in increasing order, for which  $\binom{2n}{n}$  is not divisible by 5.

$\{b_k\}_{k=1}$  is the set of all positive integers  $n$ , in increasing order, for which  $g_n$  is not divisible by 5.

$u_n$  is the number of unit digits in the base 5 representation of  $n$ .

**Theorem 1.**  $a_k$  is the number in base 5 whose digits represent the number  $k$  in base 3. If  $n \geq 1$ ,

$$\binom{2n}{n} \bmod 10 = \begin{cases} 0 & \text{if } n \notin \{a_k\} \\ \begin{matrix} 2 \\ 4 \\ 6 \\ 8 \end{matrix} & \text{if } n \in \{a_k\} \text{ and } u_n \bmod 4 = \begin{matrix} 1 \\ 2 \\ 0 \\ 3 \end{matrix} \end{cases}.$$

Note that if  $n \in \{a_k\}$ ,  $u_n$  is odd (even) if and only if  $n$  is odd (even).

*Proof.* From Lucas' theorem [1], we have

$$\binom{2n}{n} \equiv \binom{N_1}{n_1} \binom{N_2}{n_2} \cdots \binom{N_t}{n_t} \pmod{5},$$

where  $2n = (N_r \cdots N_3 N_2 N_1)_5$ ,  $n = (n_s \cdots n_3 n_2 n_1)_5$ , and  $t = \min(r, s)$ .

Suppose that for each  $i \leq t$ ,  $n_i \leq 2$ . Then, for each  $i \leq t$ ,  $N_i = 2n_i$ . Since  $n_i = 0, 1$  or  $2$ , each term of the product  $\binom{N_1}{n_1} \binom{N_2}{n_2} \cdots \binom{N_t}{n_t}$  is 1, 2 or 6. Hence,  $\binom{2n}{n}$  is not divisible by 5.

Suppose that for some  $i$ ,  $n_i > 2$ . Let  $i_m$  be the smallest value of  $i$  for which that is true. Then, if  $n_{i_m}$  is 3 or 4,  $N_{i_m}$  is 1 or 3 (resp.). In either case,  $\binom{N_{i_m}}{n_{i_m}} = 0$ , and  $\binom{2n}{n}$  is divisible by 5.

Thus,  $\{a_k\}$  is the set of all positive integers written in base 3, but interpreted as if they were written in base 5. Since  $\{a_k\}$  is in increasing order, the first part of the theorem is proved.

Suppose now that  $\binom{2n}{n}$  is not divisible by 5. Then each term of the product  $\binom{N_1}{n_1} \binom{N_2}{n_2} \cdots \binom{N_t}{n_t}$  is 1, 2 or 6 (according as  $n_i = 0, 1$ , or  $2$ ). We have, noting that  $\binom{2n}{n}$  is even,

$$\left. \begin{aligned} 2^{u_n} \bmod 10 &= 6^\dagger, 2, 4 \text{ or } 8, \\ \binom{N_1}{n_1} \binom{N_2}{n_2} \cdots \binom{N_t}{n_t} \bmod 10 &= 6, 2, 4 \text{ or } 8, \\ \binom{2n}{n} \bmod 10 &= 6, 2, 4 \text{ or } 8, \end{aligned} \right\} \text{ according as } u_n \bmod 4 = 0, 1, 2 \text{ or } 3.$$

□

**Corollary 1.1.**

$$a_k = k + 2 \sum_{i=1} \left\lfloor \frac{k}{3^i} \right\rfloor 5^{i-1}.$$

---

<sup>†</sup>Equals 1 if  $u_n = 0$ ; nevertheless, the next line follows since, if  $u_n = 0$ , at least one  $n_i$  must equal 2, making  $\binom{2n_i}{n_i} = 6$ .

*Proof.* Let  $k = (\cdots d_3 d_2 d_1)_3$ , and consider  $a_k = \sum_{i=1} d_i 5^{i-1}$ .

$$\begin{aligned} d_1 &= k - 3 \left\lfloor \frac{k}{3} \right\rfloor \\ d_2 &= \left\lfloor \frac{k}{3} \right\rfloor - 3 \left\lfloor \frac{k}{3^2} \right\rfloor \\ d_3 &= \left\lfloor \frac{k}{3^2} \right\rfloor - 3 \left\lfloor \frac{k}{3^3} \right\rfloor \\ &\vdots \quad \quad \quad \vdots \end{aligned}$$

Therefore,

$$\sum_{i=1} d_i 5^{i-1} = \sum_{i=1} \left( \left\lfloor \frac{k}{3^{i-1}} \right\rfloor - 3 \left\lfloor \frac{k}{3^i} \right\rfloor \right) 5^{i-1}.$$

Since  $\left\lfloor \frac{k}{3^i} \right\rfloor 5^i - 3 \left\lfloor \frac{k}{3^i} \right\rfloor 5^{i-1} = 2 \left\lfloor \frac{k}{3^i} \right\rfloor 5^{i-1}$ , the corollary is proved.  $\square$

**Corollary 1.2.** *Let  $\mu_k$  be the largest integer  $t$  such that  $k/3^t$  is an integer. Then,*

$$a_k - a_{k-1} = \frac{5^{\mu_k} + 1}{2}, \text{ and } a_k = 1 + \sum_{i=2}^k \frac{5^{\mu_i} + 1}{2}.$$

$\mu_k = m$  if and only if  $k \in \{j3^m\}$ , where  $j$  is a positive integer and  $j \bmod 3 \neq 0$ .

*Proof.* If  $\mu_k > 0$ , then

$$k = (\cdots d_{\mu_k+1} 0 \cdots 0)_3; \quad d_{\mu_k+1} \geq 1; \quad \text{and } k-1 = (\cdots (d_{\mu_k+1} - 1) 2 \cdots 2)_3.$$

Hence,

$$a_k - a_{k-1} = 5^{\mu_k} - 2[5^{\mu_k-1} + 5^{\mu_k-2} + \cdots + 1] = \frac{5^{\mu_k} + 1}{2}.$$

If  $\mu_k = 0$ , then

$$k = (\cdots d_1)_3; \quad d_1 \geq 1; \quad \text{and } k-1 = (\cdots (d_1 - 1))_3.$$

Hence,

$$a_k - a_{k-1} = 1 = \frac{5^{\mu_k} + 1}{2}.$$

The remaining parts of the corollary follow immediately.  $\square$

**Corollary 1.3.** *If  $k > 1$ ,*

$$a_k = \begin{cases} 5a_{\frac{k}{3}} & \text{if } k \bmod 3 = 0, \\ a_{k-1} + 1 & \text{if } k \bmod 3 \neq 0. \end{cases}$$

*Proof.* If  $k \bmod 3 = 0$ , then  $k = (\cdots d_2 0)_3$  and  $\frac{k}{3} = (\cdots d_2)_3$ . Hence,  $a_k = 5a_{\frac{k}{3}}$ .  
If  $k \bmod 3 \neq 0$ , then  $\mu_k = 0$  and from Corollary 1.2, we have  $a_k - a_{k-1} = 1$ .  $\square$

**Theorem 2.**  $b_k$  is the number in base 5 whose digits represent the number  $2k - 1$  in base 3, i.e.  $b_k = a_{2k-1}$ . Furthermore,  $g_n \bmod 10$  can only take on the values 1, 5 or 9, as follows:

$$g_n \bmod 10 = \begin{cases} 5 & \text{if } n \notin \{b_k\} \\ 1 \\ 9 \end{cases} \text{ if } n \in \{b_k\} \text{ and } u_n \bmod 4 = \begin{cases} 1 \\ 3 \end{cases}.$$

*Proof.* Let  $F(z) = \sum_n f_n z^n = \sum_n z^n \sum_i \binom{n}{i} \binom{2n-2i}{n-i}$ .

Letting  $t=n-i$ , we have

$$\begin{aligned} F(z) &= \sum_n z^n \sum_t \binom{n}{t} \binom{2t}{t} \\ &= \sum_t \binom{2t}{t} \sum_n \binom{n}{t} z^n \\ &= \frac{1}{1-z} \sum_t \binom{2t}{t} \left(\frac{z}{1-z}\right)^t \quad (\text{see [2]}) \\ &= \frac{1}{1-z} \frac{1}{\sqrt{1-\frac{4z}{1-z}}} = \frac{1}{\sqrt{1-z}} \frac{1}{\sqrt{1-5z}} \quad (\text{see [2]}) \\ &= [1 + \left(\frac{1}{4}\right) \binom{2}{1} z + \left(\frac{1}{4}\right)^2 \binom{4}{2} z^2 + \cdots] [1 + \left(\frac{1}{4}\right) \binom{2}{1} 5z + \left(\frac{1}{4}\right)^2 \binom{4}{2} 5^2 z^2 + \cdots]. \end{aligned}$$

Hence,

$$f_n = \frac{1}{4^n} \sum_{i=0}^n \binom{2i}{i} \binom{2n-2i}{n-i} 5^i,$$

and

$$g_n = \frac{1}{4^n} \sum_{i=0}^{2n} \binom{2i}{i} \binom{2n-2i}{n-i} 5^i - \binom{2n}{n},$$

$$= \frac{\sum_{i=1}^{2n} \binom{2i}{i} \binom{2n-2i}{n-i} 5^i - (4^n - 1) \binom{2n}{n}}{4^n}.$$

Thus we see that  $g_n$  is divisible by 5 if and only if  $(4^n - 1) \binom{2n}{n}$  is divisible by 5. And since  $g_n$  is odd,  $g_n \bmod 10 = 5$  if and only if  $g_n$  is divisible by 5.  $4^n - 1$  is divisible by 5 if and only if  $n$  is even. Therefore,  $g_n \bmod 10 \neq 5$  if and only if  $n$  is odd and  $n \in \{a_k\}$ . Hence,  $b_k = a_{2k-1}$ , from which it follows that  $b_k$  is the number in base 5 whose digits represent the number  $2k-1$  in base 3.

Suppose that  $g_n \bmod 10 \neq 5$ . Then  $\binom{2n}{n} \bmod 10 = c$ , where (since  $n \in \{a_k\}$  and  $n$  and  $u_n$  are odd)  $c$  is 2 or 8, according as  $u_n \bmod 4 = 1$  or 3. Thus, for some non-negative integers  $j$  and  $k$ ,  $4^n - 1 = 10j + 3$  and  $\binom{2n}{n} = 10k + c$ . Since  $\binom{2i}{i}$  is even when  $i \geq 1$ , for some non-negative integer  $q$  we have

$$4^n g_n = 10q - (10j + 3)(10k + c).$$

Since  $g_n$  is odd, and  $4^n \bmod 10 = 4$ , we have

If  $c=2$ ,  $g_n \bmod 10 = 1$ ;

if  $c=8$ ,  $g_n \bmod 10 = 9$ .

□

**Corollary 2.1.**

$$b_k = 2k - 1 + 2 \sum_{i=1}^{\lfloor \frac{2k-1}{3} \rfloor} 5^{i-1}.$$

*Proof.* This follows from Corollary 1.1, since  $b_k = a_{2k-1}$ .

□

**Corollary 2.2.** *Let  $\nu_k$  be the largest integer  $t$  for which  $\frac{(k-1)(2k-1)}{3^t}$  is an integer. Then,*

$$b_k - b_{k-1} = \frac{5^{\nu_k} + 3}{2}, \text{ and } b_k = 1 + \sum_{i=2}^k \frac{5^{\nu_i} + 3}{2}.$$

*If  $m \geq 1$ ,  $\nu_k = m$  if and only if  $k \in \left\{ \left\lceil \frac{j3^m+1}{2} \right\rceil \right\}$ , where  $j$  is a positive integer and  $j \bmod 3 \neq 0$ ; if  $m = 0$ ,  $\nu_k = m$  if and only if  $k \in \{3j\}$ , where  $j$  is a positive integer.*

*Proof.*

$$\begin{aligned} b_k &= a_{2k-1}, \\ b_k - b_{k-1} &= (a_{2k-1} - a_{2k-2}) + (a_{2k-2} - a_{2k-3}), \\ b_k - b_{k-1} &= \frac{5^{\mu_{2k-1}} + 1}{2} + \frac{5^{\mu_{2k-2}} + 1}{2}, \end{aligned}$$

where  $\mu_k$  is the largest integer  $t$  such that  $k/3^t$  is an integer.

Note that  $\nu_k$  is also the largest integer  $t$  for which  $\frac{(2k-1)(2k-2)}{3^t}$  is an integer. Then we must have one of the following cases:

$$\begin{aligned} \mu_{2k-1} &= 0 \text{ and } \mu_{2k-2} = 0, \text{ or} \\ \mu_{2k-1} &= \nu_k \text{ and } \mu_{2k-2} = 0, \text{ or} \\ \mu_{2k-1} &= 0 \text{ and } \mu_{2k-2} = \nu_k. \end{aligned}$$

In any of these cases,

$$b_k - b_{k-1} = \frac{5^{\nu_k} + 3}{2}.$$

If  $m \geq 1$ , at most one of  $(k-1)$  and  $(2k-1)$  is divisible by  $3^m$ .  $\nu_k = m$  if and only if either  $(k-1)$  or  $(2k-1)$  is divisible by  $3^m$  but not by  $3^{m+1}$ . Suppose  $m \geq 1$  and  $j \bmod 3 \neq 0$ .

$$\text{If } j \text{ is odd, } \left\lceil \frac{j3^m+1}{2} \right\rceil = \frac{j3^m+1}{2}; \quad \text{if } k = \frac{j3^m+1}{2}, \quad 2k-1 = j3^m, \text{ and } \nu_k = m.$$

$$\text{If } j \text{ is even, } \left\lceil \frac{j3^m+1}{2} \right\rceil = \frac{j3^m+2}{2}; \quad \text{if } k = \frac{j3^m+2}{2}, \quad k-1 = \frac{j3^m}{2}, \text{ and } \nu_k = m.$$

It is straightforward to show the converse, that if  $\nu_k = m \geq 1$ ,  $k \in \left\{ \left\lceil \frac{j3^m+1}{2} \right\rceil \right\}$ .

If  $m = 0$ ,  $\nu_k = m$  if and only if neither  $(k-1)$  or  $(2k-1)$  is a multiple of 3. This occurs when  $2k$  (and therefore  $k$ ) is a multiple of 3.

□

**Corollary 2.3.** *If  $k \geq 1$ ,*

$$\begin{aligned} b_{3k} &= b_{3k-1} + 2, \\ b_{3k+1} &= 5b_{k+1} - 4, & k \bmod 3 = 0, \\ &= b_{3k} + 4, & k \bmod 3 \neq 0, \\ b_{3k+2} &= 5b_{k+1}. \end{aligned}$$

*Proof.*

$$\begin{aligned} b_{3k} &= a_{6k-1} = a_{6k-2} + 1 = a_{6k-3} + 2 = b_{3k-1} + 2, \\ b_{3k+2} &= a_{6k+3} = 5a_{2k+1} = 5b_{k+1}, \\ b_{3k+1} &= a_{6k+1} = a_{6k} + 1 = 5a_{2k} + 1, \text{ and} \end{aligned}$$

if  $k \bmod 3 = 0$ ,

$$b_{3k+1} = 5(a_{2k+1} - 1) + 1 = 5b_{k+1} - 4;$$

if  $k \bmod 3 \neq 0$ ,

$$b_{3k+1} = 5(a_{2k-1} + 1) + 1 = 5b_k + 6 = b_{3k-1} + 6 = (b_{3k} - 2) + 6 = b_{3k} + 4.$$

□

**Theorem 3.**

$$f_n \bmod 10 = \begin{cases} \begin{pmatrix} 5 \\ 1 \\ 3 \\ 7 \\ 9 \end{pmatrix} & \text{if } n \notin \{a_k\} \\ \begin{pmatrix} 5 \\ 1 \\ 3 \\ 7 \\ 9 \end{pmatrix} & \text{if } n \in \{a_k\} \text{ and } u_n \bmod 4 = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 2 \end{pmatrix} \end{cases}.$$

*Proof.* Since  $f_n = \binom{2n}{n} + g_n$ , the corollary can be proved easily by combining the results of Theorem 1 and Theorem 2. □

## References

- [1] I. Vardi, Computational Recreations in Mathematica, Addison-Welsey, California, 1991, p.70 (4.4).
- [2] H.S. Wilf, generatingfunctionology (1st ed.), Academic Press, New York, 1990, p.50 (2.5.7, 2.5.11).